2. **a.** $\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4x_1^2 + 2x_2^2 + x_3^2 + 6x_1x_2 + 2x_2x_3$

b. When $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$, $\mathbf{x}^T A\mathbf{x} = 4(2)^2 + 2(-1)^2 + (5)^2 + 6(2)(-1) + 2(-1)(5) = 21.$

c. When $\mathbf{x} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $\mathbf{x}^T A\mathbf{x} = 4(1/\sqrt{3})^2 + 2(1/\sqrt{3})^2 + (1/\sqrt{3})^2 + 6(1/\sqrt{3})(1/\sqrt{3}) + 2(1/\sqrt{3})(1/\sqrt{3}) = 5.$

4. a. The matrix of the quadratic form is $\begin{bmatrix} 20 & 15/2 \\ 15/2 & -10 \end{bmatrix}$.

b. The matrix of the quadratic form is $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$.

6. **a**. The matrix of the quadratic form is $\begin{bmatrix} 5 & 5/2 & -3/2 \\ 5/2 & -1 & 0 \\ -3/2 & 0 & 7 \end{bmatrix}$.

b. The matrix of the quadratic form is $\begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = 6y_1^2 - 4y_2^2$

8. The matrix of the quadratic form is $A = \begin{bmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{bmatrix}$. The eigenvalues of A are 3, 9, and 15. An eigenvector for $\lambda = 3$ is $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$, which may be normalized to $\mathbf{u}_1 = \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$. An eigenvector for $\lambda = 9$ is $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, which may be normalized to $\mathbf{u}_2 = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$. An eigenvector for $\lambda = 15$ is $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, which may be

normalized to $\mathbf{u}_3 = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$. Then $A = PDP^{-1}$, where $P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ -2/3 & 2/3 & -1/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$ and

$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 15 \end{bmatrix}$. The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = 3y_1^2 + 9y_2^2 + 15y_3^2$$

10. The matrix of the quadratic form is $A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}$. The eigenvalues of A are 11 and 1, so the quadratic form is positive definite. An eigenvector for $\lambda = 11$ is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, which may be normalized to $\mathbf{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$. An eigenvector for $\lambda = 1$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which may be normalized to $\mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$. Then $A = PDP^{-1}$, where $P = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$ and $D = \begin{bmatrix} 11 & 0 \\ 0 & 1 \end{bmatrix}$. The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = 11y_1^2 + y_2^2$$

14. The matrix of the quadratic form is $A=\begin{bmatrix}8 & 3\\3 & 0\end{bmatrix}$. The eigenvalues of A are 9 and -1, so the quadratic form is indefinite. An eigenvector for $\lambda=9$ is $\begin{bmatrix}3\\1\end{bmatrix}$, which may be normalized to $\mathbf{u}_1=\begin{bmatrix}3/\sqrt{10}\\1/\sqrt{10}\end{bmatrix}$. An eigenvector for $\lambda=-1$ is $\begin{bmatrix}-1\\3\end{bmatrix}$, which may be normalized to $\mathbf{u}_2=\begin{bmatrix}-1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix}$. Then $A=PDP^{-1}$, where $P=[\mathbf{u}_1 \quad \mathbf{u}_2]=\begin{bmatrix}3/\sqrt{10} & -1/\sqrt{10}\\1/\sqrt{10} & 3/\sqrt{10}\end{bmatrix}$ and $D=\begin{bmatrix}9 & 0\\0 & -1\end{bmatrix}$. The desired change of variable is $\mathbf{x}=P\mathbf{y}$, and the new quadratic form is

$$\mathbf{x}^T A\mathbf{x}=(P\mathbf{y})^T A(P\mathbf{y})=\mathbf{y}^T P^T AP\mathbf{y}=\mathbf{y}^T D\mathbf{y}=9y_1^2-y_2^2$$

24. If det $A > 0$, then by Exercise 23, $\lambda_1\lambda_2 > 0$, so that λ_1 and λ_2 have the same sign; also, $ad = \det A + b^2 > 0$.

a. If det $A > 0$ and $a > 0$, then $d > 0$ also, since $ad > 0$. By Exercise 23, $\lambda_1 + \lambda_2 = a + d > 0$. Since λ_1 and λ_2 have the same sign, they are both positive. So Q is positive definite by Theorem 5.

b. If det $A > 0$ and $a < 0$, then $d < 0$ also, since $ad > 0$. By Exercise 23, $\lambda_1 + \lambda_2 = a + d < 0$. Since λ_1 and λ_2 have the same sign, they are both negative. So Q is negative definite by Theorem 5.

c. If det $A < 0$, then by Exercise 23, $\lambda_1\lambda_2 < 0$. Thus λ_1 and λ_2 have opposite signs. So Q is indefinite by Theorem 5.